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SAMPLING PROPERTIES OF COMPOSITE PERFORMANCE MEASURES AND THEIR IMPLICATIONS

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and State University

#541

College of Commerce and Business Administration
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Summary:

The statistical relationship between estimated composite performance measures and their risk proxies are derived in accordance with statistical distribution theory. It is found that the estimated composite performance measures are generally highly correlated with their risk proxies. In general, sample size, investment horizon and the market condition are three important factors in determining the degree of relationship above-mentioned. It is shown that a larger number of historical observation and an appropriate investment horizon can generally be used to reduce the sample correlation between the estimated performance measures and their risk proxy. Sampling distributions for both Sharpe and Treynor measures are also derived.

Note:

The first draft of this paper has been presented in 1978 Southern Finance Association Annual Meeting at Washington, D.C., November 9-11.

SAMPLING PROPERTIES OF COMPOSITE PERFORMANCE MEASURES AND THEIR IMPLICATIONS

I. Introduction

Following the capital asset pricing theory developed by Sharpe (1964), Lintner (1965) and Mossin (1966) [SML], Sharpe (1966), Treynor (1965) and Jensen (1968, 1969) have derived the composite performance measures--Sharpe, Treynor and Jensen performance measures for evaluating the performance of either portfolios or mutual funds. Friend and Blume (1970) [FB] have discussed the theoretical rationale of these one-parameter performance measures and their relationships by using the capital asset pricing theory. In addition, FB have also empirically shown that the risk-adjusted rates of return as measured by the composite performance measures are not successfully abstract from risk. In other words, they have found that the estimated composite performance measures are generally significantly correlated with the estimated risk proxies. Hence, FB have concluded that there exist strong biases associated with the estimated composite performance measures. Klemkosky (1973) has employed mutual funds instead of random portfolios to re-examine the biases of the estimated composite performance measures and found that there exists a relatively strong relationship between the estimated composite performance measures and estimated risk proxies. However, the possible sampling biases associated with estimated composite performance measures and their possible implications have not been carefully investigated.

The main purpose of this paper is to investigate the possible sources of bias associated with the empirical relationship between the

estimated composite performance measure and their estimated risk proxies.[1] It is shown that sample size and the investment horizon are two important factors in determining the degree of empirical relationship between estimated composite measures and estimated risk proxies. In addition, it is also shown that the above-mentioned empirical relationships are generally not independent of the market condition associated with the sample period selected for the empirical studies. In the second section, the statistical relationships between estimated Sharpe measures and its estimated risk proxies are derived. It is found that the sample size and market condition are two important factors in determining the empirical relationship between Sharpe's measure and its risk proxy. The effect of this empirical relationship on the ranking of Sharpe's performance measure is also explored. In the third section, the possible impact of investment horizon on the bias associated with testing the theoretical relationship between the Sharpe measure and its risk proxy is developed. In the fourth section, it is shown that the conclusions associated with the relationship between estimated Sharpe's measure and its risk proxy can also be extended to those of Treynor and Jensen performance measures. The implications of the statistical relationships between estimated composite performances and their risk proxies on portfolio managements are also explored. Finally, the results of the paper are summarized.

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[1] The term "bias" used in this study refers to the deviation of the empirical relationship from the theoretical relationship. Theoretically, one parameter performance measures are not expected to depend upon their risk proxies. However, it is empirically found that the estimated composite performance measures are generally highly correlated with their estimated risk proxies. To test bias associated with the capital asset pricing theory, Black, Jensen and Scholes (1972), Blume and Friend (1973), Fama and MacBeth (1973) and others have done numerous empirical studies. Most recently, Roll (1977) has carefully re-examined these empirical tests.

II. Statistical Relationship Between Estimated Sharpe Measure and Its Risk Proxy

Following Friend and Blume (1970), the theoretical relationship of the capital asset pricing model [CAPM] developed by SLM can be defined as:

$$E(R_i) - R_f = \beta_i[E(R_m) - R_f] \tag{1}$$

where R_f is the risk-free rate for borrowing or lending, R_i is the rate of return on portfolio or asset i, and R_m is the market rate of return. If the risk-free rate R_f and the index of systematic risk β_i are constant over time, then equation (1) can be rewritten in ex post or historical data as [see Jensen (1968)]:

$$R_{it} - R_f = \alpha_i + \beta_i[R_{mt} - R_f] + \varepsilon_{it} \tag{2}$$

where R_{it} is the rate of return on portfolio or asset i in period t, R_{mt} is the market rate of return in period t and ε_{it} is a random disturbance with mean zero and variance σ_ε^2 and is independent of R_{mt}. If n observations are used to estimate the parameters by ordinary least squares (OLS), equation (2) can be summed over n and averaged to obtain:

$$\bar{R}_i - R_f = \hat{\alpha}_i + \hat{\beta}_i[\bar{R}_m - R_f] \tag{3}$$

where the bar indicates an average and $\hat{\alpha}_i$ and $\hat{\beta}_i$ are least-squares estimates of α_i and β_i respectively. The estimated intercept $\hat{\alpha}_i$ is called Jensen performance measure. Assume that the standard deviation of the rates of return of portfolio i is constant over time. Then estimated Sharpe's measure can be derived from equation (3) as:

$$\frac{\bar{R}_i - R_f}{S_i} = \frac{\hat{\alpha}_i}{S_i} + \frac{[\bar{R}_m - R_f]}{S_i} \tag{4}$$

where S_i is the sample standard deviation of ex post data for portfolio i and defined as:

$$S_i = \left\{ \frac{\sum_{t=1}^{n} (R_{it} - \bar{R}_i)^2}{n} \right\}^{1/2} \tag{5}$$

If the rate of return of securities in the population (capital market) is normally distributed with mean μ and variance σ^2.[2] Then a set of sample rates of return for portfolio i can be considered as a random sample drawn from the normal population. Assume that the holding period coincides with the true investment horizon. The "investment horizon" concept will be explored in section III in detail. It follows that the sampling distribution of the average rate of return, $\bar{R}_i = \sum_{t=1}^{n} R_{it}/n$, is normally distributed with mean μ and variance σ^2/n, where n is the sample size. The random variable nS_i^2/σ^2 has a χ^2 distribution with (n - 1) degrees of freedom [see Hogg and Craig (1970)]. For simplicity, $\bar{R}$ and nS^2/σ^2 denote $\bar{R}_i$ and nS_i^2/σ^2 respectively.

To investiage the degree of relationship between the estimate Sharpe's measure and the estimated risk measure, the probability density function (p.d.f.) of the estimated risk (S) should first be derived. Following

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[2] If the holding period rates of return is lognormally distributed, then the logarithm transformation of these holding period rates of return will be normally distributed.

Hogg and Craig (1970), the p.d.f. of the random variable $nS^2/\sigma^2(y_1)$, can be defined as

$$f_1(y_1) = \begin{cases} \frac{1}{\Gamma(\frac{n-1}{2})\ 2^{(n-1)/2}} \cdot y_1^{\frac{n-1}{2} - 1} \cdot e^{-y_1/2} & \text{for } 0 < y_1 < \infty \\ 0 \text{ elsewhere} \end{cases} \quad (6)$$

let $y_2 = \frac{\sigma\sqrt{y_1}}{\sqrt{n}}$. Then $y_2 = S$. The Jacobian of the transformation is $\frac{dy_1}{dy_2} = \frac{2n}{\sigma^2} y_2$.

Thus, the p.d.f. of the estimated risk measure S is:

$$f_2(y_2) = \frac{1}{\Gamma(\frac{n-1}{2})\ 2^{(n-1)/2}} \cdot [\frac{n}{\sigma^2} y_2]^{\frac{n-1}{2} - 1} \cdot e^{\frac{ny_2^2}{2\sigma^2}} \cdot \frac{2n}{\sigma^2} y_2$$

which can be simplified as :

$$f_2(S) = \begin{cases} \frac{(\frac{n}{\sigma^2})^{1/2}}{\Gamma(\frac{n-1}{2})\ 2^{(n-3)/2}} \cdot (\frac{nS^2}{\sigma^2})^{\frac{n}{2} - 1} \cdot e^{-\frac{1}{2}(\frac{nS^2}{\sigma^2})}, & 0 < S < \infty \\ 0 \text{ elsewhere} \end{cases} \quad (7)$$

Now the covariance and the correlation coefficient between the estimated Sharpe measure and the estimated risk proxy are derived as follows:

(i) The covariance is defined as

$$\text{Cov}(\frac{\overline{R} - R_f}{S}, S)$$

$$\text{(a)} \quad = E(\overline{R} - R_f) - E(\frac{\overline{R} - R_f}{S})E(S)$$

$$\text{(b)} \quad = E(\overline{R} - R_f) \cdot [1 - E(\frac{1}{S})E(S)]$$

$$\text{(c)} \quad = d(R_f - \mu), \quad (8)$$

where $\mu = E(\bar{R})$ = expected market rate of return

$$d = E(S)E(\frac{1}{S}) - 1 = \frac{\Gamma(n/2)\Gamma(\frac{n-2}{2})}{[\Gamma(\frac{n-1}{2})]^2} - 1 > 0 \quad ^3 \tag{9}$$

$$E(\frac{1}{S}) = (\frac{n}{2\sigma^2})^{\frac{1}{2}} \cdot \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} \quad , \tag{10}$$

$$E(\frac{1}{S^2}) = (\frac{n}{2\sigma^2}) \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} \tag{11}$$

and $E(S) = (\frac{2\sigma^2}{n})^{\frac{1}{2}} \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}$ ^{4} (12)

The equality of (8b) is obtained by using the stochastic independence property between $(\bar{R} - R_f)$ and S. The independence property can be justified methodologically as follows. The joint p.d.f. of $(\bar{R} - R_f)$ and S can be derived and it can be shown that the joint p.d.f. of $(\bar{R} - R_f)$ and S can be written as a product of marginal p.d.f. of $(\bar{R} - R_f)$ and marginal p.d.f. of S.

(ii) The correlation coefficient is defined as

$$\rho(\frac{\bar{R} - R_f}{S}, S) = \frac{d(R_f - \mu)}{\sigma_{sp}\, \sigma_s} \quad ^5 \tag{13}$$

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[3] It can be shown that d converges to zero and is positive for $n > 2$. See Appendix (B) for the derivation.

[4] Equations (10), (11), and (12) are derived from equation (7). See Appendix (A) for the details.

[5] Derivation of the correlation coefficient is given in the Appendix (C).

where $\sigma_{sp}^2 = \text{Var}\left(\frac{\bar{R} - R_f}{S}\right)$

$$= \frac{\Gamma\left(\frac{n-3}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)} + (\mu - R_f)^2 \left(\frac{n}{2\sigma^2}\right) \left\{ \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right]^2 \right\} \tag{14}$$

$$\sigma_s^2 = \text{Var}\,(S) = \frac{\sigma^2}{n} \left\{ (n-1) - 2\left[\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)} \right]^2 \right\} \tag{15}$$

From equations (8) and (13), one can see that the estimated Sharpe's measure is uncorrelated with the estimated risk measure only either when the risk-free rate is equal to the mean rate, μ, of the return on the market portfolio or when sample size n is infinite. The risk-free rate is generally not equal to the mean rate of return on the market portfolio and the sample size associated with empirical work is generally finite. Therefore, the estimated Sharpe's measure is in general correlated with the estimated risk measure (S). If the risk-free rate is less than the mean rate of return on the market portfolio, the estimated Sharpe's measure is negatively correlated with the estimated risk measure. Conversely, the estimated Sharpe's measure and the estimated risk measure will be positively correlated if the risk-free rate is greater than the mean rate of return on the market portfolio.[6]

The quantity d defined in (9) is a decreasing function of the sample size n.[7] The value of d is .010 for the sample size of 50; .005 for 100. Thus the covariance defined in (8) gets smaller when the sample size

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[6] If is clear that the magnitude of the correlation coefficient, ρ is also affected by σ_{sp} and σ_s.

[7] See Appendix (B) for detail.

increases. This implies that the bias (indicated by nonzero covariance) associated with the estimated Sharpe's measure can be reduced by using a larger sample size. Hence, to reduce the bias in the empirical research, a large sample of the rates of return should be used to estimate the Sharpe measure.

The analyses derived in this section have statistically demonstrated why both Friend and Blume (1970) and Klemkosky (1973) have found that the estimated Sharpe performance measure and its risk proxy is highly correlated. Now, the sign associated with the empirical relationship between the estimated Sharpe measure and its risk proxy is discussed. As the risk-free rate proxy--Treasury bill rates and the market rates of return proxy--rates of return associated with New York Stock Exchange are measured with error, the sign associated with estimated $(R_f - \mu)$ is hardly determined.[8] Finally, it should be noted that the t statistic is generally used to test whether the estimated Sharpe's measure is significantly related with its risk proxy. The assumption of using t statistic to perform this kind of test is that the estimated Sharpe measure is normally distributed. Unfortunately, the estimated Sharpe measure is not a normal distribution but in general a noncentral t-distribution. Furthermore, the sample estimate of the standard Sharpe measure is not an unbiased estimate.[9]

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[8]The discussion of measurement errors associated with R_f can be found in Roll (1969), Friend and Blume (1970) and Lee and Jen (1978); the justification of measurement errors associated with μ can be found in Miller and Scholes (1970), Black, Jensen and Scholes (1972), Lee and Jen (1978) and Roll (1977).

[9]The distribution of estimated Sharpe's measure and the unbiased estimate of Sharpe measure are derived in Appendix (D). If $\mu = R_f$, the estimated Sharpe's measures has a central t-distribution with $(n - 1)$ degrees of freedom.

The existence of correlation between the estimated Sharpe's performance measure and the estimated risk measure will generally not affect the ranking of Sharpe's composite performance measure. However, it will affect the absolute difference among Sharpe performance measures. These can be demonstrated as follows. Let the rates of return of portfolio i be a subpopulation of the capital market having a normal distribution with mean μ_i^* and variance σ_i^{*2}. Then following equation (13), the correlation coefficient, ρ_i^*, between the estimated Sharpe's performance measure of portfolio i and the estimated risk measure, S_i, can be written as

$$\rho_i^* = \frac{d(R_f - \mu_i^*)}{\sigma_{sp}\,\sigma_s} = \frac{-d}{\sqrt{c}} \times \frac{Z_i}{a + bZ_i^2} \quad \text{for all } i \tag{16}$$

where σ_{sp}^2 and σ_s^2 are defined as in (14) and (15), respectively, with μ and σ being replaced by μ_i^* and σ_i^*, respectively:

$$\sigma_{sp} = a + bZ_i^2, \quad \sigma_s^2 = \sigma_i^2 c, \quad c > 0$$

$$a = \frac{\Gamma(\frac{n-3}{2})}{2\Gamma(\frac{n-1}{2})}, \quad b = \frac{n}{2}\left\{2a - \left[\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}\right]^2\right\},$$

$$c = \frac{1}{n}\left\{(n-1) - 2\left[\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}\right]^2\right\},$$

and $Z_i = \dfrac{\mu_i^* - R_f}{\sigma_i^*}$ = Sharpe's composite performance measure of portfolio i.

Equation (16) indicates that ρ_i^* and Z_i are nonlinearly related. The nonlinear relation between ρ_i^* and Z_i can be depicted as follows:[10]

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[10] ρ_i^* has a local maximum of $Z_i = -\sqrt{a/b}$ and a local minimum at $Z_i = \sqrt{a/b}$. Also, $\lim_{Z_i \to \pm\infty} \rho_i^* = 0$. See Appendix (E) for detail.

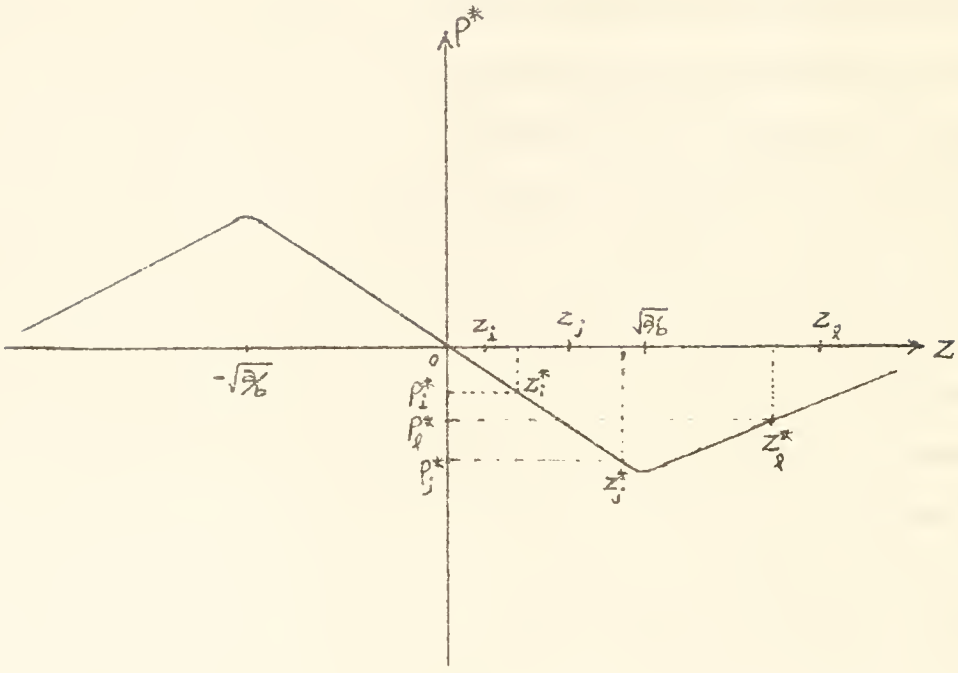

Figure 1

Figure 1 can be used to investigate the relationship between the degree of the correlation, ρ_i^*, and the ranking of Sharpe's composite performance measure. By studying Figure 1, the following conclusions can be made (without loss of generality, only portfolios with positive Z_i are considered since Sharpe's composite performance measure is mainly applied to efficient portfolios):

Figure 1 indicates that Sharpe's performance measures locate along the horizontal axis, as indicated by Z_i, Z_j, Z_ℓ, etc., if there exists no correlation between the estimated Sharpe's performance measure and the risk proxy. It also implies that Sharpe's performance measures locate along the nonlinear curve, as indicated by Z_i^*, Z_j^*, Z_ℓ^*, etc., if there exist some relationships between the estimated Sharpe's performance measure and the risk measure. In other words, if Sharpe's performance

measure is not to be affected by the correlation, ρ_{ij}^*, Z_i^*, Z_j^*, and Z_ℓ^* must equal to the corresponding Z_i, Z_j, and Z_ℓ, respectively. However, Z_i (= $(\mu_i^* - R_f)/\sigma_i^*$, subscript i may be replaced by j or ℓ) is not equivalent to Z_i^*. Z_i^* is the ex-ante Sharpe's performance measure of portfolio i whose sample estimate, $(\bar{R}_i - R_f)/S_i$, is correlated with the estimated risk measure. In other words, Z_i^* is the corresponding ex-ante performance measure that is directly estimated by $(\bar{R}_i - R_f)/S_i$. In terms of statistical concept, Z_i^* is the expected value of $(\bar{R}_i - R_f)/S_i$. It can be shown that

$$Z_i^* = E\left(\frac{\bar{R}_i - R_f}{S_i}\right) = Z_i \cdot e(n), \quad [11]$$

where $e(n) = \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{n}{\sqrt{2(n-1)}}$. Thus, $Z_i^* \neq Z_i$.

Specially, the expected value of the estimated Sharpe's measure is equal to the ex-ante Sharpe's measure of portfolio i multiplified by an error factor, e(n), which depends on sample size n. This result implies that the ranking of mutual funds (or portfolios) based on the biased estimates, $(\bar{R}_i - R_f)/S_i$, is not an unbiased ranking of mutual fund performance if different sample sizes are used for different mutual funds. This is because the error factor is not the same constant across all mutual funds (or portfolios) under evaluation. With the use of the same sample size, the ranking of mutual funds (or portfolios) based on the sample estimate, $(\bar{R}_i - R_f)/S_i$, of ex-ante Sharpe's measure provides an unbiased ranking of performance of mutual funds (or portfolios). This implies that Elton, Gruber and Padberg's (1976) simple rule of

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[11] The detailed derivation of $E[(\bar{R}_i - R_f)/S_i]$ is discussed in Appendix (D).

portfolio construction is generally not affected by substituting sample estimate of Sharpe's measure for the population representation of Sharpe's measure. However, the use of the same sample size across all mutual funds (or portfolios) cannot eliminate the absolute difference between estimated Sharpe performance measures. In other words, the absolute difference between alternative performance measures is not independent of the degree of relationship between the estimated Sharpe measure and its risk proxy. Jensen (1969) has pointed out that the performance measure can either be used to rank the performance of mutual funds (or portfolios) or be used to measure the absolute difference between the investment performance of two mutual funds (or portfolios). Therefore, the possible bias caused by the sample correlation between the estimated Sharpe's performance measure and the estimated risk proxy cannot entirely be neglected in the empirical finance research.

The behavior of the sample correlation between the estimated Sharpe's composite performance measure and the estimated risk measure and the impact of investment horizon on the sample correlation is studied in the following section.

III. Impact of Investment Horizon on Testing the Bias of Estimated Sharpe's Measure

Observation horizon used in the empirical study of portfolio management is generally not necessarily identical to the true horizon.[12] Levy (1972) has found that the portfolio with the smallest variance (or mean)

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[12]Observation horizon refers to either one day, one week, one month, one quarter or one year. The concept of "true" investment horizon implies that investors will all share the same horizon. The justification and the implication of this assumption can be found in either Lee (1976) or Levy (1972).

will have the highest performance index for an observation investment horizon being longer than the true investment horizon. The portfolio with the highest risk (and hence the highest mean) tends to have the highest performance index if an observation investment horizon is shorter than the true investment horizon. This implies that the covariance and correlation coefficient between the estimated performance index and its risk measure may well be affected if the observation investment horizon used in the empirical research does not coincide with the true investment horizon. The impact of the investment horizon on the degree of relationship between the estimated Sharpe's measure and its risk proxy is now investigated. Under the assumption of stationary returns over time and all investors having the same investment horizon, one has

$$\mu_j = \mu \text{ and } \sigma_j^2 = \sigma^2 \quad [13] \tag{17}$$

for all $j = 1, 2, \ldots, m$ where μ_j and σ_j^2 are the expected rate of return and the variance associated with j-period observations. Following Tobin (1965), one assumes stationarity over time and independence. It can be shown that

$$\tilde{E} = (1 + \mu)^m - 1 \tag{18}$$

and

$$\tilde{\sigma}^2 = [\sigma^2 + (1 + \mu)^2]^m - (1 + \mu)^{2m}, \; m > 0 \tag{19}$$

where, for the m-period case, $\tilde{E}$ and $\tilde{\sigma}^2$ are the expected rate of return and the variance, respectively, of the market portfolio. For simplicity,

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[13] μ and σ^2 have the same definitions as in the section II.

it is assumed that the true investment horizon is equal to one period of time (say, one month, one quarter, or one year, etc.). If the investment horizon assumed in empirical research does not coincide with the true investment horizon,[14] the covariance and the correlation coefficient between the estimated Sharpe's measure and the estimated risk measure are obtained as follows. Let a random sample of n rates, R_{jt}, $1 \leq t \leq n$, of return be drawn from the population. Then for the m-period case the average rate $\bar{R}_j^*$ of return is normally distributed with mean $\bar{E}$ and variance σ^2/n. The random variable nS^{*2}/σ^2 has χ^2-distribution with $(n - 1)$ degrees of freedom.[15] Following the similar analysis in the section II, the covariance and the correlation coefficient, for the m-period case, between the estimated Sharpe's measure and the estimated risk measure S* can be written as:

$$\operatorname{cov}\left(\frac{\bar{R}^* - R_f^*}{S^*}, S^*\right) = E(\bar{R}^* - R_f^*) \cdot \left[1 - E\left(\frac{1}{S^*}\right) \cdot E(S^*)\right]$$

$$= d[E(R_f^* - \bar{R}^*)]$$

$$= d[(1 + R_f)^m - (1 + \mu)^m] \quad [16] \qquad (20)$$

and $$\rho\left(\frac{\bar{R}^* - R_f^*}{S^*}, S^*\right) = \frac{d[(1 + R_f)^m - (1 + \mu)^m]}{\sigma_{sp}^* \, \sigma_s^*} \qquad (21)$$

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[14] In this case, m is not equal to 1.

[15] The sample variance S^{*2} in this section is the estimated total risk (σ^2) for the m-period case.

[16] Similar to (18), $E(R_f^*)$ can be shown as $E(R_f^*) = (1 + R_f)^m - 1$ where R_f is the risk-free rate for the one-period case (the true investment horizon).

where d is defined as in (9), R_f^* is the risk-free rate for the m-period case, and

$$\sigma_{sp}^* = \mathrm{Var}\left(\frac{\bar{R}^* - R_f^*}{S^*}\right)$$

$$= \frac{\Gamma\left(\frac{n-3}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)} + \left[(1+\mu)^m - (1+R_f)^m\right]^2 \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \left\{\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2\right\}, \quad (22)$$

$$\sigma_s^{*2} = \mathrm{Var}(S^*)$$

$$= \frac{\sigma^2}{n}\left\{(n-1) - 2\left[\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2\right\}. \quad (23)$$

It can be seen that equation (20) and (21) reduce to equations (8) and (13), respectively, if the observation horizon coincides with the true investment horizon (m = 1). Hence, an "improper" observation horizon will have impact on the degree of association between the estimated Sharpe's measure and the estimated risk measure.[17]

The impact of an improper observation horizon can be first investigated by finding the first derivative of the covariance defined as in (20) with respect to the holding period m. Thus,

$$\frac{d(\mathrm{Cov})}{dm} = d\left[(1+R_f)^m \ln(1+R_f) - (1+\mu)^m \ln(1+\mu)\right]. \quad (24)$$

---

[17] An "improper" observation horizon refers to that the time horizon used in empirical research does not coincide with the true investment horizon.

Then if R_f is greater than μ,

$$\frac{d(Cov)}{dm} > d[(1 + R_f)^m \ln(1 + R_f) - (1 + \mu)^m \ln(1 + R_f)],$$

since $\ln(1 + R_f) > \ln(1 + \mu)$ for $R_f > \mu$,

$$= d \cdot \ln(1 + R_f) \cdot [(1 + R_f)^m - (1 + \mu)^m] > 0, \quad (R_f > 0), \tag{25}$$

since the factors d, $\ln(1 + R_f)$, and $[(1 + R_f)^m - (1 + \mu)^m]$ are all positive.

This implies that the covariance between the estimated Sharpe's measure and the estimated risk measure for the m-period case is a strictly increasing function of the number of observation horizon, m, if the risk-free rate R_f is greater than the mean rate μ of return on the market portfolio for the one-period case. Similarly, if R_f is less than μ,

$$\frac{d(Cov)}{dm} < d[(1 + R_f)^m \ln(1 + \mu) - (1 + \mu)^m \ln(1 + \mu)]$$

$$= d \ln(1 + \mu)[(1 + R_f)^m - (1 + \mu)^m] < 0 \tag{26}$$

$$(\mu > R_f > 0).$$

Thus, the covariance for the m-period case is a strictly decreasing function of the number of observation horizon, m, if R_f is less than μ. The graphic representation of the relationship between the m-period covariance and the number of observation horizon, m, is shown as follows:

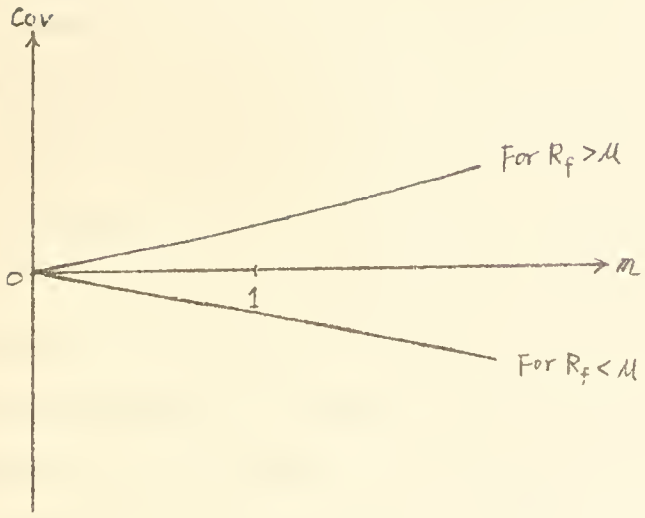

Figure 2

The impact of an improper observation horizon on the m-period covariance can be summarized as follows.

(i) A longer observation horizon than the true investment horizon.

For the risk-free rate R_f being greater than the mean rate μ of return on the market portfolio, the m-period covariance becomes larger in the positive magnitude by moving the holding period m away from the true investment horizon-one period. Similarly, for R_f being less than μ the m-period covariance increases its magnitude negatively when the observation horizon is longer than the true investment horizon. Hence, if a longer observation horizon than the true investment horizon is used, the m-period covariance is expected to increase in either positive or negative magnitude depending on the order of the magnitudes of the risk-free rate R_f and the mean rate μ of return on the market portfolio.

(ii) A shorter observation horizon than the true investment horizon.

As the observation horizon is shorter than the true investment horizon, the m-period covariance becomes smaller if the risk-free rate R_f is greater than the mean rate of return on the market portfolio. And a weaker negative covariance is expected for the case where R_f is less than μ if the observation horizon approaches zero.

Therefore, in conducting empirical research, an improper observation horizon will have impact on the covariance between the estimated Sharpe's measure and the estimated risk measure. It is interesting to note that a shorter observation horizon than the true investment horizon will reduce the dependence of the estimated Sharpe's performance measure on its estimated risk measure. A longer observation horizon than the true investment horizon will magnify the dependence. These results indicate that a shorter observation horizon should be used in the empirical research to reduce the bias associated with the estimated Sharpe's measure.

IV. Sampling Properties of Treynor and Jensen Measures

In the sections II and III, it is shown that the estimated Sharpe's measure is generally highly correlated with the estimated risk proxy. It is also shown that the degree of relationship between the estimated Sharpe's measure is generally affected by the sample size, the market condition, and the investment horizon.

In this section, it is demonstrated that the conclusions associated with the relationship between estimated Sharpe's measure and its risk proxy can be extended to those of Treynor's and Jensen's performance measures. The covariance between estimated Treynor's measure and its estimated risk measure is first explored.

Under the assumption of normality security return, theory of least squares has indicated that the estimated beta coefficient, $\hat{\beta}$, and the sample mean excess rate of return, $\bar{R} - R_f$, are independently normally distributed.[18] This follows that the estimated Treynor's measure, $(\bar{R} - R_f)/\hat{\beta}$, is a ratio of two independent normal variables. Fieller (1932) has shown that the ratio of two normal variables does not have finite moments. To find the covariance between $(\bar{R} - R_f)/\hat{\beta}$ and $\hat{\beta}$, we may assume that $\hat{\beta}$ takes values in some positive range.[19] It eliminates the existence of infinite moments. Then, the truncated distribution of the estimated risk measure, $\hat{\beta}$, can be written as

$$f(\hat{\beta}) = \begin{cases} \frac{1}{k} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\beta}}} \cdot \exp\left[-\frac{1}{2}\left(\frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}}\right)^2\right], \\ \qquad \text{for } a \leq \hat{\beta} \leq b,\ a > 0 \\ 0 \quad \text{elsewhere} \end{cases} \tag{27}$$

where $\sigma_{\hat{\beta}}^2 = \mathrm{Var}(\hat{\beta}) = \dfrac{\sigma_\varepsilon^2}{\sum_{t=1}^{n} (R_{mt} - \bar{R}_m)^2}$,

$\beta = E(\hat{\beta})$, the expected value of non-truncated $\hat{\beta}$,

and $k = \int_a^b \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{\beta}}} \cdot \exp\left[-\frac{1}{2}\left(\frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}}\right)^2\right] d\hat{\beta}$ = a constant

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[18] Roll (1977) has shown that the sample mean return is an exact linear function of $\hat{\beta}$ only if a proxy (ex-post) of the market portfolio is a sample efficient portfolio.

[19] For example, $\hat{\beta}$ may be assumed to take values between 0.0001 and 10. This will guarantee the existence of finite moments.

Since $\hat{\beta}$ and $(\bar{R} - R_f)$ are independently distributed, the truncated distribution of $\hat{\beta}$ will also be independent of the distribution of $(\bar{R} - R_f)$. With this independent property, the covariance between the estimated Treynor's measure and its estimated risk measure can be easily obtained as follows:[20]

$$\text{Cov}\left(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}\right) = E(\bar{R} - R_f) - E(\hat{\beta})E\left(\frac{\bar{R} - R_f}{\hat{\beta}}\right)$$

$$= E(\bar{R} - R_f) - \beta^{+} \cdot E(\bar{R} - R_f)E\left(\frac{1}{\hat{\beta}}\right),$$

by using independent property

$$= (\beta^{+}C - 1)[E(R_f - \bar{R})]$$

$$= (\beta^{+}C - 1)(R_f - \mu), \tag{28}$$

where $C = E\left(\frac{1}{\hat{\beta}}\right)$ is a constant defined in equation (G-5) of Appendix (G),

β^{+} = the expected value of the truncated $\hat{\beta} = \beta + \ell$,[21]

$$\ell = \frac{1}{k}\left[Z\left(\frac{a - \beta}{\sigma_{\hat{\beta}}}\right) - Z\left(\frac{b - \beta}{\sigma_{\hat{\beta}}}\right)\right],$$

and $$Z\left(\frac{a - \beta}{\sigma_{\hat{\beta}}}\right) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(a - \beta)^2}{2\sigma_{\hat{\beta}}^2}\right].$$

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[20] The distribution function of the estimated Treynor's measure can be easily obtained from Fieller's (1932) results. See Appendix (E) for derivation.

[21] See Johnson and Kotz (1970) for the derivation of the expected value of a truncated normal distribution.

Equation (28) is identical to equation (8) except the scalar term. Similarly, the correlation coefficient between the estimated Treynor's measure and its estimated risk measure can be expressed as

$$\rho\left(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}\right) = \frac{(\beta^{+}C - 1)\cdot(R_f - \mu)}{\sigma_T \, \sigma_{\hat{\beta}}} \tag{29}$$

where σ_T denotes the standard deviation of the estimated Treynor's measure with the truncated value of $\hat{\beta}$.[22] Equation (29) is also identical to (13) except the scalars, $(\beta^{+}C - 1)$, σ_T and $\sigma_{\hat{\beta}}$. Furthermore, following the analysis of the impact of investment horizon in Section III, the m-period covariance and correlation coefficient between the estimated Treynor's measure and the estimated risk measure can be easily written as, respectively,

$$\mathrm{Cov}\left(\frac{\bar{R}^* - R_f^*}{\hat{\beta}^*}, \hat{\beta}^*\right) = (\beta^{+}C - 1)\left[(1 + R_f)^m - (1 + \mu)^m\right] \tag{30}$$

and

$$\left(\frac{\bar{R}^* - R_f^*}{\hat{\beta}^*}, \hat{\beta}^*\right) = \frac{(\beta^{+}C - 1)\left[(1 + R_f)^m - (1 + \mu)^m\right]}{\sigma_T^* \, \sigma_{\hat{\beta}}^*} \tag{31}$$

where asterisks denote the m-period sample estimtes (or parameters, σ_T^* and $\sigma_{\hat{\beta}}^*$). Thus, equations (30) and (31) are identical to equations (20) and (21) except scalar constants. Therefore, equations (28) through (31) implies that the conclusions associated with the relationship between

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[22] The explicit form of σ_T is not necessary for the analysis.

estimated Sharpe's measure and its risk measure can also be extended to the estimated Treynor's measure and its estimated risk measure.

However, it is important to note that the estimated Treynor's measure is a biased estimator of ex-ante Treynor's measure:[23]

$$E\left(\frac{\bar{R} - R_f}{\hat{\beta}}\right) = \left(\frac{\mu - R_f}{\beta}\right) \cdot e_\beta, \tag{32}$$

where $e_\beta = \dfrac{\beta}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\dfrac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \cdot \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{(i-j)} 2^{(j+1)/2}\, \Gamma\left(\frac{j+1}{2}\right) m_j\right]$, (33)

$$m^* = (a' + b'),$$

$$a' = \frac{a - \beta}{\sigma_{\hat{\beta}}}, \quad b' = \frac{b - \beta}{\sigma_{\hat{\beta}}},$$

and

$$m_j = \int_{a'^2}^{b'^2} \frac{x^{\frac{(j+1)}{2} - 1}\, e^{-x/2}}{\Gamma\left(\frac{j+1}{2}\right) 2^{(j+1)/2}}\, dx$$

= a chi-square probability (integral).

The biased factor, e_β, associated with the expected value of the estimated Treynor's measure depends on the value of a mutual fund's (or portfolio's) systematic risk. The biased factor will vary from one mutual fund (or portfolio) to another because of different systematic risk associated with different mutual fund (or portfolio). Therefore, the ranking of mutual

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[23] See Appendix (G) for the derivation.

fund performance using the biased estimate, $(\bar{R} - R_f)/\hat{\beta}$, of Treynor's measure will not be an unbiased ranking of ex-ante Treynor's measure, $(\mu - R_f)/\beta$. In addition, the absolute difference between two estimated Treynor's measures will also be affected by the biased factor, e_β. Note that the bias factor cannot be used to obtain an unbiased Treynor's measure.

Now, the statistical relationship between estimated Jensen's measure and its estimated risk proxy is examined.

Following Heinen (1969) and the definitions defined in this paper, the covariance between estimated Jensen's measure and its estimated risk proxy can be defined as

$$\mathrm{Cov}(\hat{\alpha}_i, \hat{\beta}_i) = \frac{-\bar{\chi}_m \; S_\varepsilon^2}{\sum_{t=1}^{n} (\chi_{mt} - \bar{\chi})^2} \tag{34}$$

$$\text{where } \chi_{mt} = R_{mt} - R_f, \; \bar{\chi}_m = \bar{R}_m - R_f \tag{35}$$

S_ε^2 is the residual variance associated with equation (2). Equation (34) indicates that the estimated Jensen performance measure is generally highly correlated with its estimated risk proxy. This relationship has been found by both Friend and Blume (1970) and Klemkosky (1973). It is clear that the degree of relationship can also be affected by the sample size, market condition and the investment horizon.

The results of this section and previous sections have shown that the estimated composite performance measures can be highly correlated with its estimated risk proxies. In general, sample size, investment horizon and the market condition are three important factors in determining the

degree of the above-mentioned relationships. Johnson and Burgess (1975) and Burgess and Johnson (1976) have investigated the effects of sample sizes and sampling fluctuation on the accuracy of both portfolio and security analyses. They have concluded that the number of historical observation is important to yield efficient portfolio performance characteristics. Their conclusion is similar to the result associated with the impact of sample size on the relationship between estimated composite measures and risk proxies derived in this study.

V. Summary

In this paper, the statistical relationships between estimated one parameter composite performance measures and their risk proxies are derived in accordance with statistical distribution theory. It is found that the above-mentioned statistical relationships are generally affected by sample size, investment horizon and the market condition associated with the sample period selected for empirical studies. In addition, it is shown that large historical observations and an appropriate investment horizon can generally be used to improve the usefulness of composite performance measures in both portfolio and mutual fund managements. Finally, it is shown that the standard sample estimate of both Sharpe and Treynor measures are not unbiased estimators and the unbiased estimates of Sharpe measure is also derived.

APPENDIX

(A) Derivation of equation (10), (11), and (12):

$$E(\frac{1}{S}) = \int_0^\infty \frac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2})\ 2^{(n-3)/2}} (\frac{1}{S})\ (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-1}\ e^{\frac{-1}{2}(\frac{nS^2}{\sigma^2})}\ ds$$

$$= C_1 \int_0^\infty (\frac{n}{\sigma^2})^{\frac{n}{2}-1}\ S^{n-3}\ e^{\frac{-1}{2}(\frac{nS^2}{\sigma^2})}\ ds$$

where $C_1 = \dfrac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2})\ 2^{(n-3)/2}}$

$$= \frac{C_1}{2} \cdot \int_0^\infty (\frac{nS^2}{\sigma^2})^{\frac{n}{2}-2}\ e^{-(\frac{nS^2}{2\sigma^2})}\ d(\frac{nS^2}{\sigma^2})$$

$$= \frac{C_1}{2} \int_0^\infty Z^{\frac{n-2}{2}-1}\ e^{-Z/2}\ dZ,$$

where $Z = \dfrac{nS^2}{\sigma^2}$

$$= \frac{1}{2} \cdot \frac{(\frac{n}{\sigma^2})^{\frac{1}{2}}}{\Gamma(\frac{n-1}{2})\ 2^{(n-3)/2}} \cdot \Gamma(\frac{n-2}{2})\ 2^{(\frac{n-2}{2})}$$

$$= (\frac{n}{2\sigma^2})^{\frac{1}{2}} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} \quad .$$

$$E\left(\frac{1}{S^2}\right) = \int_0^\infty C_1 \frac{1}{S^2} \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2} - 1} e^{\frac{-1}{2}\left(\frac{nS^2}{\sigma^2}\right)} ds$$

$$= C_1 \int_0^\infty (S^2)^{-1} \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2} - 1} e^{\frac{-1}{2}\left(\frac{nS^2}{\sigma^2}\right)} \cdot \frac{\sigma^2}{2nS} d\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{C_1}{2} \left(\frac{n}{\sigma^2}\right)^{\frac{1}{2}} \int_0^\infty \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n-3}{2} - 1} e^{-(nS^2/\sigma^2)/2} d\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{1}{2} \frac{\left(\frac{n}{2}\right)^{\frac{1}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{(n-3)/2}} \cdot \Gamma\left(\frac{n-3}{2}\right) 2^{\frac{n-3}{2}}$$

$$= \frac{n}{2\sigma^2} \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

And

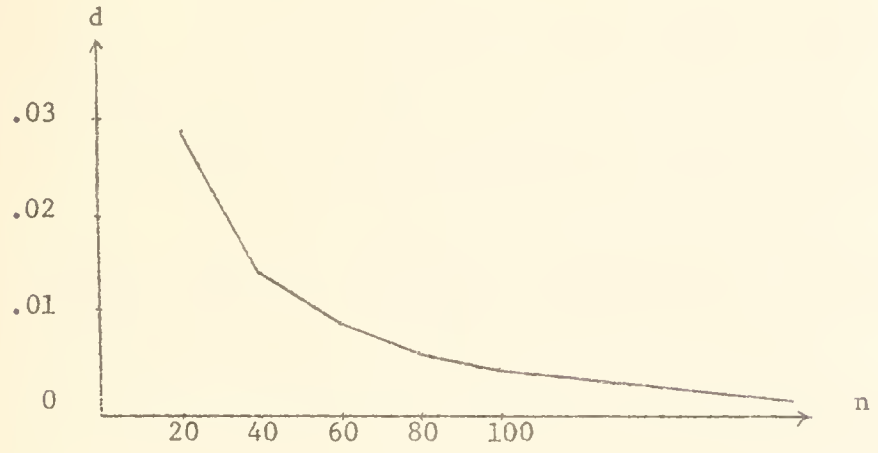
$$E(S) = \int_0^\infty C_1 S \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2} - 1} e^{\frac{-1}{2}\left(\frac{nS^2}{\sigma^2}\right)} ds$$

$$= C_1 \int_0^\infty \frac{\sigma^2}{2n} \left(\frac{nS^2}{\sigma^2}\right)^{\frac{n}{2} - 1} e^{\frac{-1}{2}\left(\frac{nS^2}{\sigma^2}\right)} d\left(\frac{nS^2}{\sigma^2}\right)$$

$$= \frac{\sigma^2 C_1}{2n} \cdot \Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}} = \left(\frac{2\sigma^2}{n}\right)^{\frac{1}{2}} \frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}$$

(B) $$d = \frac{\Gamma(\frac{n}{2})\ \Gamma(\frac{n-2}{2})}{[\Gamma(\frac{n-1}{2})]^2} - 1:$$

The value of d is evaluated by digital computer for all values of n, $2 < n < 500$. The computed values of d indicate that d is a positve and decreasing function of n. The curve of d is depicted as follows:



Some values of d are tabulated as follows:

n	20	40	60	80	100	120	150	210	340	480
d	.028	.013	.009	.006	.005	.004	.003	.002	.0015	.0010

(C) Show that

$$\rho(\frac{\overline{R} - R_f}{S},\ S) = \frac{d(R_f - \mu)}{\sigma_{sp}\ \sigma_s}$$

where d is defined as in (9), σ_{sp} and σ_s defined as in (14) and (15), respectively.

Proof:

$$\sigma_{sp}^2 = \mathrm{Var}\left(\frac{\bar{R} - R_f}{S}\right) = E\left[\left(\frac{\bar{R} - R_f}{S}\right)^2\right] - \left[E\left(\frac{\bar{R} - R_f}{S}\right)\right]^2$$

$$= E(\bar{R} - R_f)^2 \cdot E\left(\frac{1}{S^2}\right) - [E(\bar{R} - R_f)]^2 \cdot \left[E\left(\frac{1}{S}\right)\right]^2, \qquad \text{(a)}$$

using the property of independence between $\bar{R}$ and S.

$$= [\sigma^2/n + (\mu - R_f)^2] \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - (\mu - R_f)^2 \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2,$$

by substituting equations (10) and (11) in equation (a)

$$= \frac{\Gamma\left(\frac{n-3}{2}\right)}{2\Gamma\left(\frac{n-1}{2}\right)} + (\mu - R_f)^2 \cdot \left(\frac{n}{2\sigma^2}\right) \cdot \left[\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} - \left[\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2\right],$$

which is equation (14)

And

$$\sigma_s^2 = \mathrm{Var}(S) = E(S^2) - [E(S)]^2$$

$$= \frac{n-1}{n} \cdot \sigma^2 - \frac{2\sigma^2}{n}\left[\frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2$$

since $E(S^2) = \frac{n-1}{n}\sigma_s^2$ by Hogg and Craig (1970)

and E(S) is obtained as in equation (12).

$$= \frac{\sigma^2}{n}\left\{(n-1) - 2\left[\frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}\right]^2\right\}$$

which is equation (15).

Hence

$$\rho\left(\frac{\bar{R} - R_f}{S}, S\right) = \frac{d(R_f - \mu_R)}{\sigma_{sp}\,\sigma_s} \quad .$$

(D) Determine the distribution of the estimated Sharpe's performance measure:

Since $(\bar{R}-R_f)$ is normally distributed with mean $\mu - R_f$ and variance σ^2/n, $(\bar{R} - R_f)/(\sqrt{\sigma^2/n})$ is normally distributed with mean $(\mu - R_f)/\sqrt{\sigma^2/n}$ and variance 1. By the stochastic independence of the average excess rate of return and the estimated risk measure, the estimated Sharpe's performance measure can be rewritten as

$$\frac{\bar{R} - R_f}{S} = \frac{(\bar{R} - R_f)/\sqrt{\frac{\sigma^2}{n}}}{\sqrt{\frac{nS^2/\sigma^2}{n-1}}} \times \frac{1}{\sqrt{n-1}}$$

$$= t'(\delta) \times \frac{1}{\sqrt{n-1}} \quad ,$$

where $t'(\delta) = \dfrac{(\bar{R} - R_f)/\sqrt{\frac{\sigma^2}{n}}}{\sqrt{\frac{nS^2/\sigma^2}{n-1}}}$ is a noncentral t-distribution with non-centrality parameter $\delta[= (\mu - R_f)/\sqrt{\sigma^2/n}]$. [See Hogg and Craig (1970)]. Thus, the estimated Sharpe's performances measure is a noncentral t-distribution. If $\mu = R_f$, the distribution of the estimated Sharpe's performance reduces to a central t-distribution with $(n - 1)$ degrees of freedom (since $\delta = 0$).

Then, Rahatgi (1976) has shown that

$$E[t'(\delta)] = \delta \cdot \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\frac{n}{2}}, \quad n > 1.$$

Based upon this result, an unbiased estimator for the Sharpe performance measure can be derived by substituting equation (34) into equation (33), i.e.,

$$E(\frac{\bar{R} - R_f}{S}) = [\delta \cdot \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\frac{n}{2}}] \cdot \frac{1}{\sqrt{n-1}}$$

$$= (\frac{\mu - R_f}{\sigma}) \cdot \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{n}{\sqrt{2}} \cdot \frac{1}{\sqrt{n-1}}$$

$$= (\frac{\mu - R_f}{\sigma}) \cdot e(n),$$

where $e(n) = \dfrac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \cdot \dfrac{n}{\sqrt{2(n-1)}}$

The above equation implies that the sample estimate of Sharpe measure is not an unbiased estimator if the security rates of return is normally distributed.

Thus, an unbiased estimator of Sharpe's performance measure can be defined as

$$\left(\frac{\overline{R} - R_f}{S}\right)\left[\frac{1}{e(n)}\right].$$

(E) Show that ρ_i^* has a local maximum and minimum:

$$\frac{d\rho_i^*}{dZ_i} = \frac{-d}{\sqrt{C}} \times \frac{(a + bZ_i^2) - Z_i(2bZ_i)}{(a + bZ_i^2)^2} = \frac{-d}{\sqrt{C}} \times \frac{a - bZ_i^2}{(a + bZ_i^2)^2} = 0$$

This implies that $Z_i = \pm \sqrt{a/b}$, $a > 0$, $b > 0$.

$$\frac{d^2\rho_i^*}{dZ_i^2} = \frac{-d}{\sqrt{C}} \times \frac{(a + bZ_i^2)^2(-2bZ_i) - (a - bZ_i^2) \cdot 2(a + bZ_i^2)(2bZ_i)}{(a + bZ_i^2)^4}$$

$$= \frac{+d}{\sqrt{C}} \times \frac{2bZ_i(3a - bZ_i^2)}{(a + bZ_i^2)^2}$$

Thus, if $Z_i = \sqrt{a/b}$,

$$\frac{d^2\rho_i^*}{dZ_i^2} = \frac{+d}{\sqrt{C}} \times \frac{2b(\sqrt{a/b})(3a - a)}{(2a)^3} = \frac{d}{\sqrt{C}} \times \frac{4ab(\sqrt{a/b})}{8a^3} > 0$$

This implies that ρ_i^* has a local minimum at $Z_i = \sqrt{a/b}$. Similarly, ρ_i^* has a local maximum at $Z_i = -\sqrt{a/b}$. Finally,

$$\lim_{Z_i \to \infty} \rho_i^* = \lim_{Z_i \to \infty} \left(\frac{-d}{\sqrt{C}} \times \frac{Z_i}{a + bZ_i^2}\right) = \lim_{Z_i \to \infty} \left(\frac{-d}{\sqrt{C}} \times \frac{1}{2bZ_i}\right) = 0$$

Similarly, $\lim_{Z_i \to -\infty} \rho_i^* = 0$.

(F) Fieller (1932) has obtained a marginal density function of a ratio of two normal random variables. The distribution function of the estimated Treynor's measure can be obtained from Fieller's equation (24) by letting the correlation coefficient be zero:

$$\psi(\nu) = \frac{1}{\pi} \cdot \frac{\sigma_1 \sigma_2}{\sigma_2^2 + \nu^2\sigma_1^2} \cdot e^{-\frac{1}{2}\left[\frac{\beta^2}{\sigma_1^2} + \frac{(\mu - R_f)^2}{\sigma_2^2}\right]}$$

$$+ e^{-\frac{1}{2}\frac{[(\mu - R_f) - a\beta]^2}{\sigma_2^2 + \nu^2\sigma_1^2}} \times \frac{-\beta\sigma_2^2 - \nu\sigma_1^2(\mu - R_f)}{\pi(\sigma_2^2 + \nu^2\sigma_1^2)^{3/2}}$$

$$\times \int_0^h e^{-\frac{1}{2}\mu^2} d\mu , \qquad -\infty < \nu < \infty,$$

where $\nu = \dfrac{\overline{R} - R_f}{\hat{\beta}}$,

$$\sigma_1^2 = \mathrm{Var}(\hat{\beta}) = \frac{\sigma_\varepsilon^2}{\sum_{t=1}^{n} (R_{mt} - \overline{R}_m)^2} ,$$

and $\sigma_2^2 = \mathrm{Var}(\overline{R} - R_f) = \dfrac{\sigma^2}{n}$ $(\sigma = \mathrm{Var}(R_{it}))$.

(G) The derivation of $E(\frac{\bar{R} - R_f}{\hat{\beta}})$:

By independent property of $(\bar{R} - R_f)$ and truncated $\hat{\beta}$,

$$E(\frac{\bar{R} - R_f}{\hat{\beta}}) = E(\bar{R} - R_f)E(\frac{1}{\hat{\beta}}) = (\mu - R_f)\cdot E(\frac{1}{\hat{\beta}}). \tag{G-1}$$

Thus, $E(1/\hat{\beta})$ is determined as follows.

$$E(\frac{1}{\hat{\beta}}) = \int_a^b \frac{1}{\hat{\beta}} \cdot \frac{1}{k\sqrt{2\pi}\sigma_{\hat{\beta}}} \cdot \exp[-\frac{1}{2}(\frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}})^2]d\hat{\beta} \tag{G-2}$$

Let $y = (\hat{\beta} - \beta)/\sigma_{\hat{\beta}}$. The Jacobian of the transformation is $|d\hat{\beta}/dy| = \sigma_{\hat{\beta}}$. Thus, equation (G-2) becomes

$$E(\frac{1}{\hat{\beta}}) = \frac{1}{k\sqrt{2\pi}} \int_{a'}^{b'} (\frac{1}{y\sigma_{\hat{\beta}} + \beta})e^{-y^2/2}\, dy, \tag{G-3}$$

where $a' = (a - \beta)/\sigma_{\hat{\beta}}$ and $b' = (b - \beta)/\sigma_{\hat{\beta}}$. (Here, $a < \beta$.) To integrate the integrand in (G-3), it is necessary to express the function, $g(y) = 1/(y\sigma_{\hat{\beta}} + \beta)$, in terms of an infinite series. Since $1/(y\sigma_{\hat{\beta}} + \beta)$ is defined for all values of y in the interval (a', b'). Note that the point, $y = -\beta/\sigma_{\hat{\beta}}$, at which the function $g(y)$ is undefined is not in (a', b'). The Taylor's series expansion of $g(y)$ at a positive point, say $m^* = (a'+b')$,** is

$$g(y) = \frac{1}{y\sigma_{\hat{\beta}} + \beta} = \frac{1}{(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [-\frac{\sigma_{\hat{\beta}}(y-m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}]^i, \tag{G-4}$$

where $a' \leq y \leq b'$.

It can be shown that $|\frac{-\sigma_{\hat{\beta}}(y-m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}| < 1$ as follows:

**b is chosen such that $m^* > 0$, i.e., $(a + b) > 2\beta$.

$$\frac{-\sigma_{\hat{\beta}}(y-m^*)}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{-\sigma_{\hat{\beta}}y + m^*\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{(m^*\sigma_{\hat{\beta}} + \beta) - \hat{\beta}}{m^*\sigma_{\hat{\beta}} + \beta}, \quad \text{since } \sigma_{\hat{\beta}}y = \hat{\beta} - \beta < 1,$$

since $\hat{\beta} > 0$ and $-\hat{\beta} < 0$. (It is the truncated $\hat{\beta}$.)

Also,

$$\frac{-\sigma_{\hat{\beta}}(y-m^*)}{m^*\sigma_{\hat{\beta}} + \beta} > \frac{-\sigma_{\hat{\beta}}y}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{-\hat{\beta}+\beta}{(a+b) - \beta}, \text{ since } m^*\sigma_{\hat{\beta}} + \beta = (a+b) - \beta > 0$$

$$> \frac{-\hat{\beta}}{(a+b) - \beta} > \frac{-\hat{\beta}}{(a+b)} > -1,$$

since β is assumed to be positive in the truncated case and $\hat{\beta}<(a+b)$.

This implies that

$$\left|\frac{-\sigma_{\hat{\beta}}(y - m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}\right| < 1 \quad \text{for} \quad a' \leq y \leq b'.$$

Since every term of the infinite series in (G-4) has an absolute magnitude less than one, the series converges uniformly and absolutely by advanced calculus. Then, equation (G-3) can be rewritten as

$$E\left(\frac{1}{\hat{\beta}}\right) = \frac{1}{k\sqrt{2\pi}} \int_{a'}^{b'} \frac{1}{(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[-\frac{\sigma_{\hat{\beta}}(y - m^*)}{m^*\sigma_{\hat{\beta}} + \beta}\right]^i \cdot e^{-y^2/2}\, dy$$

$$= \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \int_{a'}^{b'} \sum_{j=0}^{i} \binom{i}{j} y^j (-m^*)^{i-j} e^{-y^2/2} dy\right],$$

since the series converges uniformly.

Let $x = y^2$. Then $y = \pm\sqrt{x}$ and $dy/dx = \pm \frac{1}{2} x^{-1/2}$. Thus,

$$E\left(\frac{1}{\tilde{\beta}}\right) = \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j}\right].$$

$$\int_{a'^2}^{b'^2} x^{\frac{(j+1)}{2} - 1} e^{-x/2} \, dx$$

$$= \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \cdot \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j} 2^{(j+1)/2} \Gamma\left(\frac{j+1}{2}\right) m_j\right] \quad \text{(G-5)}$$

where $m_j = \int_{a'^2}^{b'^2} \frac{x^{\frac{(j+1)}{2} - 1} e^{-x/2}}{\Gamma\left(\frac{j+1}{2}\right) 2^{(j+1)/2}} \, dx.$

Hence, substituting equation (G-5) into equation (G-1) leads to

$$E\left(\frac{\bar{R} - R_f}{\hat{\beta}}\right) = \left(\frac{\mu - R_f}{\beta}\right) \cdot e_\beta \quad \text{(G-6)}$$

where $$e_\beta = \frac{\beta}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j}\right].$$

$$2^{(j+1)/2} \Gamma\left(\frac{j+1}{2}\right) m_j]. \quad \text{(G-7)}$$

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